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The Role of Schemes in Solving Word Problems

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Abstract: *This paper is about schemes and how they can assist in the learning of word problems in mathematics. First the paper presents the theoretical background and working definitions for schemes suggested by cognitive psychology. Analysis of various types of word problems and the developmental trend that children exhibit in solving these problems follows. A special analysis is devoted to a number of constrained schemes that underlie common mathematical word problem types. The use of schemes is extended to open-end word problems showing that these too can be better solved methodically with the help of schemes.*

Theoretical background

This paper deals with schemes[1] and how schemes assist in learning arithmetic word problems. Many philosophers and psychologists considered the notion of schemes with some variations. The term 'scheme' is used as a means of perceiving the world as an innate logical development and as patterns of action.

Piaget (Piaget & Inhelder, 1969; Piaget, 1970; Piaget, 1971, 1967; Piaget, 1985) dealt with schemes of action, writing:

> *A schema of an action consists in those aspects which are repeatable, transposable, or generalisable* (Piaget, 1980, *p.205).*

Fischbein (Fischbein & Grossman, 1997; Fischbein 1999) based his definition on Piaget´s notion of a scheme which defines a scheme not merely as a perceptual framework, but rather as a pattern of action. Fischbein in particular believed that a scheme is also a strategy for solving a certain class of problems. He too, stressed the behavioral aspect of a scheme. For him a scheme is a plan for action. Let us take a simple example (a kinetic scheme of action): Opening a door by its handle, knowing that the handle is to be pushed down and then the door pushed in or pulled out, is a scheme. We hardly ever pay attention to it because for us it is instinctive, but once we enter another system which we do not recognize, such as in a train

[1] We will use the term 'scheme' and 'schemes' (in English), although 'schema', 'schemata' (in Latin) and 'schemas' are used in quotations from other authors, and we regard all these to be interchangeable.

(pushing the green knob to open the door), we need to construct a new scheme of action.

Fischbein distinguished between two types of schemata: the first "indicates a kind of condensed, simplified representation of a class of objects or events" (Fischbein, 1999, p.36) and the second is "adaptive behavior of an organism …achieved by assimilation and accommodation" (Fischbein, p.37). For Fischbein, a schema is a program which enables the individual to (a) record, process, control and mentally integrate information, and (b) react meaningfully and efficiently to the environmental stimuli.

Cognitive psychologists of the 80's considered schemes to be semantic nets expressing relations, or scripts of behavior, such as well known behavior at a birthday party or at a restaurant (Schank & Abelson, 1977; Anderson, 1980). Within the stream of cognitive psychology of the 80's we can find the following description of what a scheme is:

> *A schema is a mental representation of some aspect of the world. It has slots that are related to each other in prescribed ways and that are filled by stimuli to create an instantiation of the schema…*(Howard, 1987, p. 176)

Rumelhart and Norman (1985) characterized schemes as follows:

> *Schemas are data structures for representing the generic concepts stored in memory. There are schemas for generalized concepts underlying objects, situation, events, sequences of events, action and sequences of actions. …Schemas in some sense represent the stereotypes of these concepts. Roughly, schemas are like models of the outside world. To process information with the use of schema is to determine which model best fits the incoming information (p. 35-36).*
>
> *Some important features of schemas:*
> 1. *Schemas have variables;*
> 2. *Schemas can embed, one within another;*
> 3. *Schemas represent knowledge at all levels of abstraction;*
> 4. *Schemas represent knowledge rather than definitions;*
> 5. *Schemas are active recognition devices whose processing is aimed at the evaluation of their goodness of fit to the data being processed* (p. 36).

Although a scheme is composed of many details, it is not merely a pile of objects, but rather an organized collection of objects with the relations between them giving meaning to all its components. Thus, when observing the components below (Figure 1), they seem to hardly have any meaning, yet within the scheme 'face', they are readily and fully understood (Figure 2). (Taken from Rumelhart, 1980)

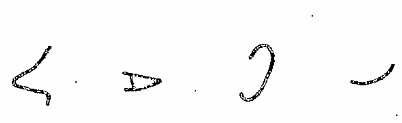

Figure 1.

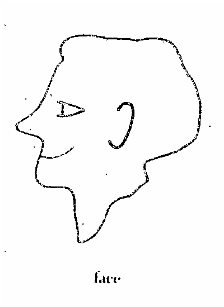

Figure 2.

Schemes and Word Problems

Difficulties encountered by students in solving word problems are cognitively based and, to some extent, universal. For the past thirty years researchers have been inquiring into what lies behind the major difficulties children encounter with word problems. At one stage, researchers made a distinction between different additive problems and classified them into three main categories: The research was conducted in several countries and all agreed on the same categorization of additive word problems (Nesher & Teubal, 1975; Nesher & Katriel, 1977; Carpenter, Moser, & Romberg, 1982; Vergnaud, 1982; Nesher, 1982a; Nesher, Greeno & Riley, 1982b; Greeno & Kintsch, 1985; Riley & Greeno, 1988; Vergnaud, 1988; Verschaffel, 1993; Kintsch, 1994). Table 1 presents the main categories of additive word problems agreed upon:

Further research on the level of difficulty of each type has detected a more subtle distinction between problems within each category, as presented in Table 2.

Table 1.
Three general semantic categories of addition and subtraction word problems

Current Name of Category	Characteristics	Example
1. Combine	Involves a static relationship between sets. Asks about the Union set or one of two disjoint subsets	There are 3 boys and 4 girls. How many children are there altogether?
2. Change	Describes an increase or decrease in some initial state, to produce a final state.	John has 6 marbles. He lost 2 marbles. How many marbles does John have now?
3. Compare	Involves static comparison between two sets. Asks about the difference set or about one of the sets where the difference set is given	Tom has 6 marbles. Joe has 4 marbles. How many more marbles does Tom have than Joe?

As can be seen, each of the 14 problems presents a different level of difficulty to the solvers. Change 5 and 6 problems, in which the Final and Change sets were given and the initial set was unknown, were most difficult at all grade levels and in all studied countries. As with the Change problems, the difficulty of Combine and Compare problems also varied depending on the unknown. Combine 2 problems, for example, were significantly more difficult than Combine 1 problems. Compare problems in which the referent was unknown were more difficult than any of the other Compare problems.

The Development of Schemes

Schemes develop. As noted by Piaget, this is achieved by two main mechanisms: assimilation and accommodation. As children start to describe the world with numbers, they form the notion of a set (Greeno 1978; Nesher et al., 1982b; Fuson 1992). Their first mathematical step will be counting objects. When the children start school, we can assume that they have already attained the following Level 1 schemes:

Table 2.
14 types of addition and subtraction word problems (semantic categories and position of the unknown)

Title	General Description	Percent[2] of success (%)
Combine 1	Question about the union set (whole).	79 - 86
Combine 2	Question about one subset (part).	46 - 52
Change 1	Increasing, question about the final set.	79 - 82
Change 2	Decreasing, question about the final set.	72 - 75
Change 3	Increasing, question about the change.	62 - 72
Change 4	Decreasing, question about the change.	75 - 77
Change 5	Increasing, question about the initial set.	28 - 48
Change 6	Decreasing, question about the initial set.	39 - 49
Compare 1	Mentioning 'more', question about the difference set.	76 - 85
Compare 2	Mentioning 'less', question about the difference set.	66 - 75
Compare 3	Mentioning 'more', question about the 'compared'.	65 - 80
Compare 4	Mentioning 'less', question about the 'compared'.	66 - 81
Compare 5	Mentioning 'more', question about the referent.	43 -60
Compare 6	Mentioning 'less', question about the referent.	35 - 54

At **Level 1*:*** Children have already constructed the schemes for counting (which means the ability for **Predication and Cardinality)**. They are able to identify sets by a variety of verbal descriptions (concept names, locations, points of time, possessions, etc.); they have the ability to perform simple operations such as adding or removing objects from sets and understanding that it changes the number of the objects in the set. Their arithmetic competence consists of the ability to count and find the cardinal number of a given set (for details, see Nesher et al., 1982b). When in the possession of these kinds of schemes, children can solve various types of problems, counting each time from the beginning.

At **Level 2*: C***hildren are able to **link** events by cause and effect and anticipate results of actions described in ordinary language. We say that they have constructed a **change scheme.** In arithmetic, the + and - operations are seen as distinct, not related, and the = sign (equality sign) is understood as an instruction to perform a procedure.

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[2] There were variations among researchers that are described as a range of successes rather than a single number

At **Level 3*:*** Children are able to integrate a "**Part-Part-Whole" scheme** that can be used to represent set relations with a slot for an unknown quantity, for a set that was defined by its concepts (predicates). A set can also be induced by means of relative comparison. The schemes at this level are related to the understanding of class inclusion. In arithmetic at this level, the additive structure is reversible and includes the = sign as denoting an equivalent relation. The underlying scheme for this level is schematically described in Table 3.

Table 3.
Aspects of Development

Scheme Level	Empirical Knowledge	Mathematical Operations
1 Counting sets	Reference to sets, adding and removing members of sets. Understanding 'put', 'give', 'take', etc. as denoting change in location or possession	Ability to count and find the cardinal number of a set. The order among numbers. $2 < 5 < 8$
2 Change	Ability to link events by cause and effect. Reference to the amount of change. Understanding a sequence of events ordered in time in a non-reversible manner	Understanding addition and subtraction as procedures. '+' and '-'are distinct $a + b \rightarrow c$ $a - b \rightarrow c$
3 Part-Part-Whole	A reversible part-part-whole schema is available, and can be used to find the unknown part in any slot in a sequence of events. Understanding class-inclusion.	Understanding the relation among three numbers in an equation (=). Connection between addition and subtraction: if $a + b = c$, then $c - b = a$ and $c - a = b$
4 Directional relations	Reversibility of non-symmetrical relations. Ability to handle directional description (more/ less), and quantify a relational set (relative comparison).	The ability to handle inequality and its relationship to equality, equalizing it by addition or subtraction: if $a > b$, then $a - c = b$ and $b + c = a$

Note that the Part-Part-Whole scheme of Level 3 is reversible and also incorporates the arithmetic additive relationship which now includes the operations + and - as related, serving as inverse operations operating on the same structure.

At **Level 4***:* Children can employ the reversible scheme for non-symmetrical relations (that was already initiated at Level 2). Directional (ordered) descriptions (i.e., 'more', 'less') can be handled in a flexible fashion. The arithmetic at this level includes the ability to handle inequality, and the ability to equalize inequality by addition or subtraction.

In the description of the above developmental levels we assume that at least two distinct structures of knowledge are involved:

(a) A child's knowledge of the world, and
(b) A child's knowledge of logico-mathematical structures.

The sources of these two knowledge structures, as was noted by Piaget (Piaget, 1970; Piaget, 1971,1967), are not the same. The logico-mathematical growth of children cannot, of course, be understood as divorced from their experience with physical objects, yet the mechanism for that growth is different, as indicated by Piaget's reference to 'simple abstraction' and 'reflective abstraction' (Piaget & Inhelder 1969; Piaget 1971, 1967).

Understanding the Levels of Performance in Solving Arithmetic Word Problems

In the last section we outlined the general kinds of knowledge that we assume underlie arithmetic problem-solving. We turn now to the empirical findings and show how they can be understood in the light of the above developmental levels. Table 3 presents this development in two distinct knowledge realms: the empirical knowledge about the world and the logico-mathematical knowledge.

We will now explain the relation between Table 3 and the ability to solve word problems. In Table 3, Level 1 is defined by the ability to represent and operate on single sets. The knowledge available to represent information about sets includes schemes for identifying sets and the ability to represent the cardinality of a set (Riley, 1983; Riley & Greeno, 1988). These schemes are sufficient to solve Change 1 and 2 and Combine 1 problems, which share two main characteristics: (1) the strategy required for solving the problem can be selected on the basis of partial and local information, and (2) the solution set is directly available for counting at the time the question is asked.

For example, consider how Level 1 children could solve Combine 1 problems such as:

> *Joe has 3 marbles.*
> *Tom has 5 marbles.*
> *How many marbles do Joe and Tom have together?*

Understanding the first sentence requires that children use their knowledge of the possessive verb 'has', to represent a set of marbles belonging to Joe. On the basis of this representation, children then select an appropriate display and count out a set of three objects. This procedure is repeated for the second sentence. To determine the answer, children need only to count the combined set. Thus solving Combine 1 problems involves three isolated actions of counting well-defined sets. Similarly, Change 1 and 2 problem**s** can also be solved on the basis of local problem features that specify completely separate actions of counting.

For example, consider the following Change 1 problem:

> *Joe had 3 marbles.*
> *Then Tom gave him 5 more marbles.*
> *How many marbles does Joe have?*

The first sentence of this problem is identical to the first sentence of Combine 1. The second sentence requires that children first understand that 'gave' in this case refers to an increase, and then increase the initial set by the appropriate number of marbles. The answer again involves counting the set described by the question by counting it all, as a separate assignment.

In contrast, consider what happens when the solution set cannot be determined by reference to the final ownership alone, as in Change 3*:*

> *Joe had 3 marbles.*
> *Then Tom gave him some more marbles.*
> *Now Joe has 8 marbles.*
> *How many marbles did Tom give Joe?*

Solving this problem involves counting out an initial set of 3 marbles, then increasing that set by 5 marbles in response to 'Now Joe has 8 marbles'. At this point, Level 1 children's representation of the problem cannot cope with a missing set and they relate to the final set of marbles belonging to Joe (which is mentioned) as the set of added marbles. Therefore when asked, 'How many marbles did Tom give to Joe?' children answer, 'eight', or ˙eleven´ and not the correct answer, '5' (Riley, 1983).

Thus, our analysis not only explains how Level 1 children solve certain problems successfully, but also why children at that level fail to solve other problems which require the ability to link events (as in the case of Change 3 problems). This is the knowledge that we attribute to children at Level 2. We will not discuss here in detail the other levels that appear in the table. The interested reader can find these in Nesher et al. (1982b).

The ability to solve problems such as Change 5 and Change 6 at Level 3 introduces one of the most powerful predictions of our theoretical model. In these problems, the semantic schemes that originated in children's experiences with ordinary language *contradict* the newly learned semantics of addition and subtraction.

For example, let us consider a problem of the Change 5 type:

Dan had some marbles.
He found 5 more marbles.
Now he has 8 marbles.
How many marbles did he have to start with?

Children's experience with natural language will direct them to add ('found' means 'having more', thus 'adding'). Choosing to subtract (for the correct solution) can be achieved only if the semantics of natural language and the mathematical language are differentiated as two autonomous systems, so that each can be further elaborated to reach the necessary coordination between the two systems. Solving a Change 5 problem involves interpreting the 'initial state', the 'change' and the 'final state' of the above problem in a non-temporal manner as in a part-part-whole relationship. Since one part and the whole are given, finding the second part is achieved by subtraction.

Thus, at Level 3 children are able to create the mapping between their natural language knowledge and mathematical knowledge, *not* on the basis of isolated verbal cues, but rather, on the basis of understanding the underlying semantics of both languages. Now children are able to impose the logico-mathematical structure, which is reversible and atemporal, on a sequential-temporal situation described by natural language. To sum up, our hypothesis concerning the developmental levels explains the type of problems that can be solved by children at a given level, as shown in Table 4.

While we have shown in detail the evolvement of the additive scheme, other studies similarly describe the development of the multiplicative scheme (Davydov, 1969; Fischbein, Deri, Nello & Marino, 1985; Nesher, 1988; Greer, 1994; Schwartz, 1995).

Table 4.
Word Problems by Levels of Development

Type of Problem	Level 1	Level 2	Level 3	Level 4
Combine 1	X			
Combine 2			X	
Change 1	X			
Change 2	X			
Change 3		X		
Change 4		X		
Change 5			X	
Change 6			X	
Compare 1			X	
Compare 2			X	
Compare 3			X	
Compare 4			X	
Compare 5				X
Compare 6				X

More complex mathematical structures

Once children have attained the autonomous mathematical additive or multiplicative structure, it will serve them in *all* contexts where addition, subtraction, multiplication or division is required. A three-argument scheme (for those operations) can be depicted in a diagram consisting of three related components (see Figure 3). The nature of one-step word problems is such that the text describes two of the components (needed for binary operations) in full, and the solver is asked to discover the quantity of a third component which is given only by its set description. Equipped with a scheme for the whole situation, each component can be assigned a role (a part or a whole, a product or a factor). In the diagram the two upper boxes represent the subsets (parts, or factors), while the bottom box represents the union of the subsets (the whole, or the product).

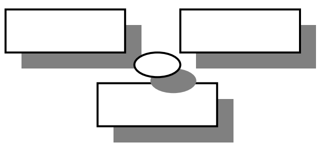

Figure 3.

With the above scheme encapsulated as a mathematical object we can construct higher mathematical hierarchies that will serve us as schemes in more complex situations. The following schemes demonstrate, for example, the possible situations for two-step problems (see Figure 4).

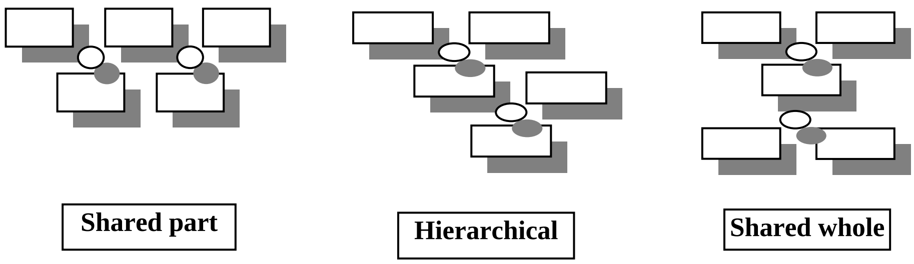

Figure 4.

Note that these three schemes exhaust all possibilities of combining the basic schemes as described in Figure 3. One main advantage of using schemes is the fact that the schemes we use in mathematics, while limited in number, correspond to many situations in which they can be applied. This should direct us to teach mathematics via general schemes.

The condensed nature of the schemes can be described by the following example: Problems 1 to 4 are all derived from the same situation that could be described by the *Hierarchical* scheme but each time we ask regarding a different component (See Figure 5).

Problem 1: *A total of 35 flowers are distributed equally among 7 vases. In each vase are 3 tulips and the rest are roses. How many roses are there in each vase?*

Problem 2: *A total of 35 flowers are distributed equally among vases. In each vase are 3 tulips and 4 roses. How many vases are there?*

Problem 3: *Flowers are distributed equally among 7 vases. In each vase are 3 tulips and 4 roses. How many flowers are there in all vases?*

Problem 4: *A total of 35 flowers are distributed equally among 7 vases. In each vase are 4 roses and the rest are tulips. How many tulips are there in each vase?*

All above problems share the same structure, since it is the same situation.

Problems 5 is a situation described by the "**Shared Whole**" scheme:

<u>Problem 5</u>: *There are 8 girls and 12 boys in the classroom. They were divided into 5 equal groups. How many children were in each group?*

The following example illustrates the case of the **"Shared Part"** scheme:

<u>Problem 6</u>**:** *17 children came to the party. At the end of the party, 15 flowers were left and they were given to the girls. Each girl got 3 flowers. How many boys were at the party?*

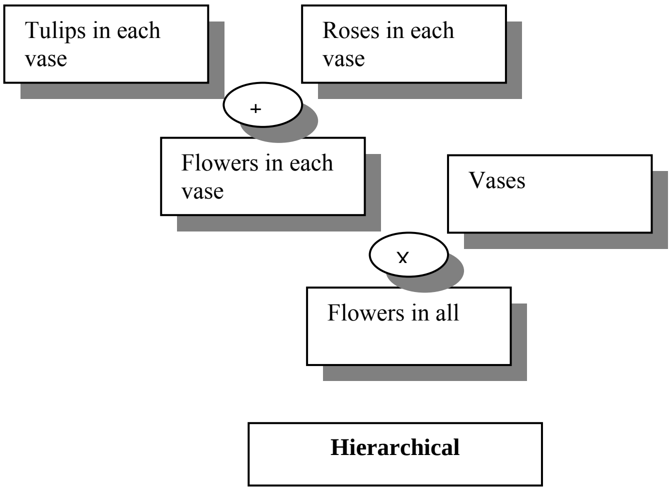

Figure 5.

Problems 5 and 6, of course, can be elaborated into other problems emerging from the same situation, as detailed in Problems 1 to 4, which actually describe the same situation. The schemes enable us to show the basic mathematical structures underlying the parsimony of word problems beyond the variety of contexts and textual formulations.

The distinction between schemes and flowcharts

We would like to emphasize that the scheme in our interpretation is a condensed encapsulation of a situation and not a flowchart. Some researchers have confounded a flowchart for action and a scheme. For example, Reusser (1988; 1992) has a similar graphical net describing the solution path of a problem. His graphical notation is similar to ours but does not include the notion of schema. Let us examine the two problems in detail:

> <u>*Problem 7:*</u> *At the party were 12 boys, at the end of the party 15 flowers were distributed to the girls, each girl got 3 flowers. How many children were there at the party?*

> <u>*Problem 8*</u>: *17 children came to the party. 12 were boys and the rest were girls. At the end of the party each girl got 3 flowers. How many flowers were distributed to all the girls?*

Reusser would set up two separate flowcharts as follows: each pair of boxes is combined by the next operation required. Thus, the flowchart can be constructed only after already knowing how to solve the problem and how to order the given information in a sequence that fits the solution path. The flowchart for Problem 7 is given in figure 6 and for Problem 8 in figure 7.

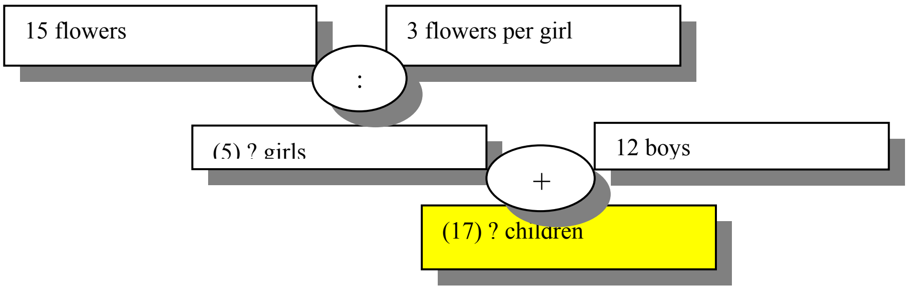

Figure 6.

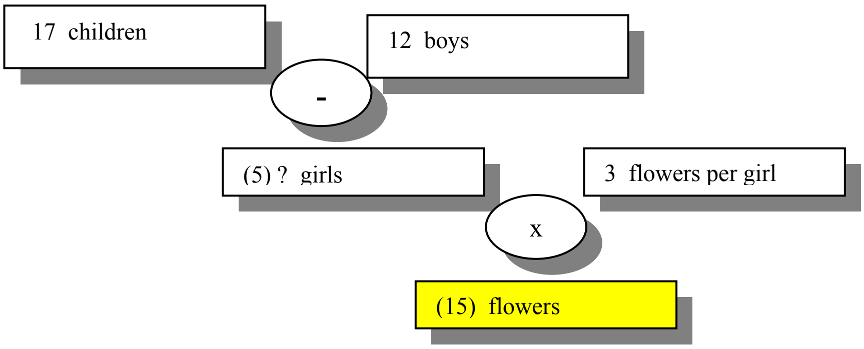

Figure 7.

If a scheme approach is adopted, then the two problems are derived from a single schema, the Shared-Part Schema**,** in which the known and unknown quantities are given for each problem in different slots of the same scheme. Thus, Problems 6, 7 and 8 will have the same scheme (See figure 10).

Another attempt was made by Schwartz (1986) who suggested a graphical net, showing the operational connection between quantities given in the problems. For him, too, the net is a flowchart which is different for each of the above two problems:

Schwartz's net for Problem 7 is given in Figure 8:

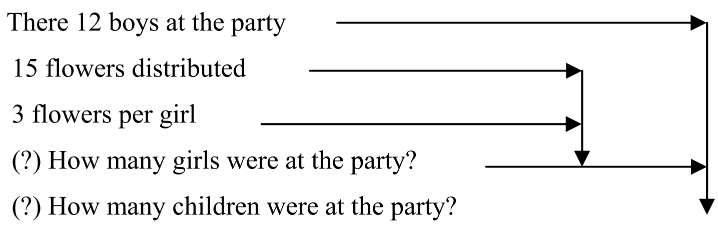

Figure 8.

Schwartz's net for Problem 8 is given in Figure 9:

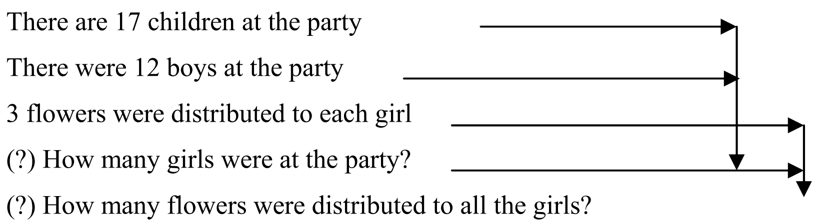

Figure 9.

As can be seen again, the table can constructed only one already realizes how to solve the problem. He should know on which quantities to operate first and on which later. The role of a scheme is different (see figure 10). It is the same scheme for problems 6, 7, and 8.

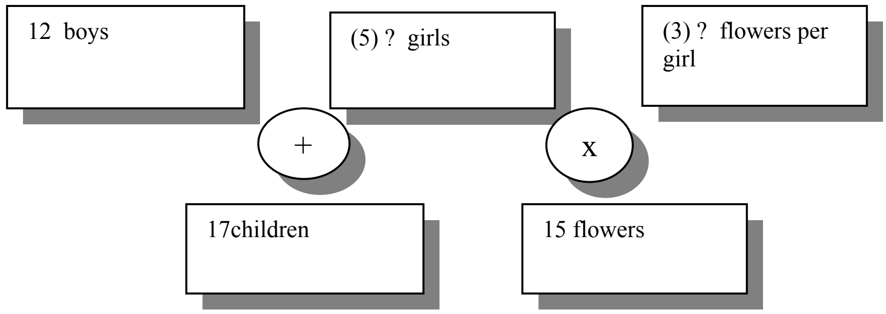

Figure 10.

In SPA, the role of the students is to construct a scheme for the situation, rather than for the solution path. The solution path is automatically derived from the scheme of the situation and the uncompleted slots. When a scheme is formulated for the entire situation then both problems are derived from the same situation and the same scheme. Thus, where Reusser and Schwartz present a new flowchart for each problem, the schemes in our case condense the multiplicity of two-step problems, into only three possible basic schemes.

The above analyses were tested empirically (for further details see Hershkovitz, Nesher & Yerushalmy, 1990; Hershkovitz & Nesher, 1996; Hershkovitz & Nesher, 1997). Since both Schwartz's approach and the schemes approach were designed as software programs (the AP for Schwartz's approach and the SPA for the schemes approach, respectively) an experiment was conducted comparing success in solving two-step word problems using the two software programs, with compelling results:

While there was no difference between the two groups examined in respect to easy problems (we used a prior grading for problem difficulty from another empirical study (Nesher & Hershkovitz, 1994)), the difference emerged in two types of cases: first, the scheme approach was critical for difficult problems, and second, it was most helpful for low achievers (see Table 5).

Moreover, we noticed another difference regarding the mode of solving word problems. While solvers using the Schwartz AP program for difficult problems were playing blindly and rapidly with the given numbers, mostly erroneously, solvers who were using the schemes could not rush, as they had to first fill in the scheme before doing anything with the numbers. Thus, although it took them longer to analyze the problems, the process in most cases ended with a correct solution.

Table 5
Attainment scores for easy and difficult problems by low and high achievers

Students	Program	Easy problems	Difficult problems
All	SPA	1.78	1.74
	AP	1.72	0.97
Low achievers	SPA	1.67	1.56
	AP	1.40	0.53
High achievers	SPA	1.86	1.86
	AP	2.00	1.35

This was particularly true for the low achievers. We thus realized that the scheme approach has introduced a new style of coping with the difficulties of word problems. Instead of rushing to do something, anything, with the available numbers, or sink into despair, we observed a sincere effort to solve the problem correctly and invest more than just one second in analyzing the situation.

The following diagram (Figure 11) shows the difference in style while working on two-step word problems with AP and SPA. The diagram shows (in seconds) three stages of work on the problems: (1) reading the text; (2) analysis of the text and (3) working on the solution.

AP	Low achievers	Reading	****
		Analyzing	**************
		Solving	************
	High achievers	Reading	*****
		Analyzing	**********************
		Solving	********************
SPA	Low achievers	Reading	********
		Analyzing	****************************
		Solving	****************
	High achievers	Reading	*******
		Analyzing	*****************************
		Solving	***********

* means 10 seconds

Figure 11.

As can be seen from the diagram, low achievers and high achievers are behaving similarly when using the SPA software, while the low-achievers working with AP are devoting a short attention span to the problems, and of course failing to solve it correctly in the end. More interesting is the fact that working with schemes caused the low achievers to gain better scores than the high achievers working with AP (see Table 5).

Can Schemes be Taught?

Schemes are considered by all to be innate and abstract. Yet it seems that we can foster construction of schemes by being aware of their role. One way, suggested by Dorfler (1991), is to work with students on the graphical representation of the schemes:

> *An image schemata thereby is a schematic structure which in a highly stylized form depicts or exhibits the main feature and relationships of situations and processes to which potentially the word refers". (pp. 19)…." Image schemata are used to make relationships cognitively manipulable and understandable (p. 20). I want to view an image schema as the perceptive and/or cognitive interaction 'with the just imagined'. I will call the latter 'concrete carrier. The representations are more or less suited for stimulating the process (p. 21).*

The image and its graphical representation can serve as "concrete carriers" that enable using the scheme with its inner relations and connections. We regard the graphical representation of the schemes as a very powerful tool on which the students practice how to analyze a given word problem in terms of the abstract underlying scheme.

Evidence of the fact that in any problem solving situation there are underlying schemes that direct the solution which can be learned from our following experiment: A group of 49 fifth and sixth graders were asked to read aloud each of 3 two-step problems, then to retell (repeat) the problem and only then solve it (for details see Hershkovitz & Nesher, 2001). An example of one such problem is Problem 9:

> *Problem 9:* *Lunch bags were prepared for the children going on a trip. Each lunch bag included 5 pieces of fruit, 2 of which were apples and the rest dates. For all lunch bags they needed 240 dates. How many lunch bags were prepared altogether?*

Protocols of the three problems were analyzed along the three dimensions exhibited by the solvers:

1. Exact retelling of the problem.

2**.** Changing the story, but retaining the scheme of the story (either by adding details about the content of the trip, or changing the order of the given numerical information).

For example Yoni noted*:*

> *Somebody from the school sport committee, or somebody else, I am not sure who he was, has prepared lunch bags for a trip organized by the sport club. In each bag they put 5 pieces of fruit that included 2 apples and 3 dates.*

He then wrote “2+3=5”, and continued:

> *In order to prepare all the lunch bags they used 240 dates. How many children received lunch bags?* He wrote “240:3=80”, adding *now I also know how many apples they used,* and wrote again “80 x 2=160".

3. Changing the story by changing the underlying scheme.

For example, Sharon wrote:

> *They prepared lunch bags. There were 5 pieces of fruit in each bag, 3 dates and 2 apples;* wrote “240:5=48”, and said, “*240 are all the pieces of fruit and each child receives 5 pieces of fruit".*

He has thus simplified the problem into a one-step division problem without solving the original problem.

The main finding of this study was that almost all children who correctly solved the problems elaborated the original story into a situation familiar to them, while conserving the original scheme. All students who erred in solving the problems changed the story into a simplified version that allowed them to use a more elementary arithmetic scheme.

General schemes and open problems

To teach by using schemes means to teach by means of the most generalized cases. Let us use two different problems as examples:

> *Problem 10:* *Roni visited a farm. He saw cows and chickens. He did not remember how many were there, but he remembered the guide saying that they had altogether 100 legs. How many cows and how many chickens were there?*

There is of course more than one answer to this question. One can guess at least one or two answers, but the interesting process is to discover all possible answers, as well as to answer some other questions such as:

1. Is it possible to have the same number of legs for cows and chickens?
2. Is it possible to have the same number of heads for cows and chickens?
3. Is it possible to have an odd number of chickens?
4. What is the minimum number of chickens, if there are both chickens and cows in the farm? What is the maximum number?
5. What are the possibilities, if there are more cows than chickens?
6. Make up your own questions….

Anyone who has a general scheme for this type of problem, such as in Figure 12, can easily comprehend all possible alternatives and answer many different questions.

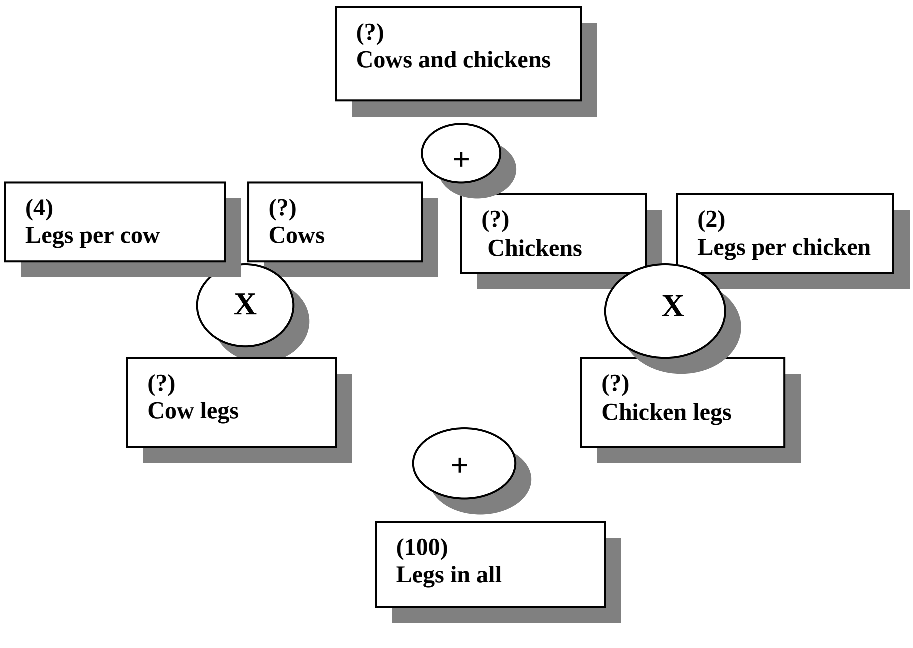

Figure 12.

Problem 11: *Two cars are traveling from Jerusalem to visit another city. Car B left 3 hours after car A.*

1. Will they meet on their way? Under what conditions?
2. When will they meet?

3. How far from Jerusalem will they meet?
4. What would be the description of two cars traveling toward each other?
5. Make up your own questions…

A graphical scheme for problem 11 can be seen in Figure 13.

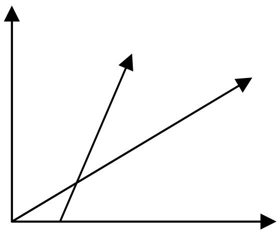

Figure 13.

In a similar way one can construct a general scheme for cars that are traveling in opposite directions and meet somewhere on the way, and we can use the same schemes for other contexts such as work, voltage, and so on.

Once we have generated a representation for a general scheme for such problems, we can learn that each problem in our standard textbooks is just one such case among many others, that the singular cases are not important, and that it is the general scheme that counts.

Some concluding remarks

We have tried to demonstrate that the ability to solve problems in mathematics is dependent on the level of schemes and structures available to the children and these change due to time and to learning. Students can benefit most if aware of the schemes that are needed at each level of learning and present the problems in their most general form.

We believe that the goal of teaching mathematics in general and word problems in particular, is to enrich the repertoire of schemes available to each student. Patterson (Patterson & Smith, 1986) says that experts in a given area have rich and complex schemes that enable them to absorb new information in that area and suggest the most efficient solution. Similarly Lester (Lester & Garofalo, 1982; Lester, 1994) who tried to characterize good solvers, indicated that the knowledge of "good" solvers is within a knowledge base and is organized by rich schemes. The collection of word problems that textbooks present at school is usually large, but not always directed by the study of schemes. We believe that we can foster our students' problem solving ability in mathematics by enriching their mathematical schemes as the building blocks of the students' cognition.

It would be proper to remember that the notion of scheme did not start with Piaget, or within mathematics education research, but rather can be traced to Kant (1724-1804) who, in a chapter on The Schematism of the Pure Concepts of Understanding, wrote:

> *These conditions of sensibility constitute the universal condition under which alone the category can be applied to any object. This formal and pure condition of sensibility to which the employment of the concept of understanding is restricted, we shall entitle the schema of the concept. The procedure of understanding in these schemata we shall entitle the schematism of pure understanding. The schema is in itself always a product of imagination. Since, however, the synthesis of imagination aims at no special intuition but only at unity in the determination of sensibility, the schema has to be distinguished from the image." (Kant 1980 Edition, p. 182)*

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